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ON THE ORIENTABILITY OF THE ASSET EQUILIBRIUM MANIFOLD

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Abstract This paper addresses partly an open question raised in the Handbook of Mathematical Economics about the orientability of the pseudo-equilibrium manifold in the basic two-period General Equilibrium with Incomplete markets (GEI) model. For a broad class of explicit asset structures, it is proved that the asset equilibrium space is an orientable manifold if $S-J$ is even. This implies, under the same conditions, the orientability of the pseudo-equilibrium manifold. By a standard homotopy argument, it also entails the index theorem for $S-J$ even. A particular case is Momi's result, i.e the index theorem for generic endowments and real asset structures if $S-J$ is even.

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1 Introduction

Consider the General Equilibrium with Incomplete markets (GEI) model with two periods, S states of nature at the second period, J assets, L commodities and a finite number of consummers. The asset structure V is a smooth mapping assigning a $S\times J$ matrix to each normalized price vector $p\in P:=\{(p_1,...p_L)\in \mathbb{R}^L_{++}, p_L=1\}$.

Little is known about the asset equilibrium space[2]

$$\mathcal{A}:=\{(p,E)\in P\times G^J(\mathbb{R}^S),\ \mathrm{span}V(p)\subset E\}.$$

It has been shown to be a smooth manifold (see [14] or [9]) for generic real asset structures[3]. Besides, it is clearly a manifold homeomorphic to P if one has

[1]I wish to thank an anonymous referee for valuable comments. Errors are mine.

[2]In the following, $G^J(\mathbb{R}^S)$ denotes the set consisting of all J-dimensional linear subspaces of $\mathbb{R}^S$.

[3]In fact, it is proved that the asset equilibrium space, parameterized with a real asset structure, is a manifold. It easily follows, from Sard Theorem, that $\mathcal{A}$ is a manifold for generic real asset structures.

rank$V(p) = J$ for every $p \in P$, which is true, for example, in the case of nominal or numeraire asset structures. More generally, given a fixed and non constant rank asset structure, most of the issues about the topological structure of $\mathcal{A}$ remain open.

However, the structure of $\mathcal{A}$ is of primary importance in the GEI model. In particular, it is closely related to the structure of the pseudo-equilibrium manifold $\mathcal{PE}$, defined by

$$\mathcal{PE} := \{(w, p, E) \in \Omega \times \mathcal{A}, Z(w, p, E) = 0\} \tag{1}$$

where Ω denotes the set of all agents' endowments and $Z : \Omega \times \mathcal{A} \to \mathrm{I\!R}^L$ denotes the pseudo-equilibrium aggregate excess demand of the economy[4] (see [8]).

An important open issue is the orientability of $\mathcal{PE}$. In the Handbook of Mathematical Economics ([11]), one can read p.1553 : "It is not known, in general, if $\mathcal{PE}$ is an orientable manifold....$\mathcal{PE}$ will certainly be orientable if $V(p)$ always has full rank, and an index theorem could be written out for this case." But in the case when $V(p)$ can change rank with $p \in P$, "...then two problems arise in attempting to verify if $\mathcal{PE}$ is orientable. The construction of $\mathcal{PE}$ in Duffie-Shafer simply shows that $\mathcal{PE}$ can be locally represented as a solution of a transverse system of equations, from which it is difficult to obtain information about orientability. Secondly, $G^J(\mathrm{I\!R}^S)$ itself is orientable if and only if S is even, although it is difficult to believe that being able to write down an index formula should depend on the parity of S, which is not of immediate economic significance."

Actually, we shall see that, under standard assumptions, the orientability of $\mathcal{A}$ implies the orientability of $\mathcal{PE}$.

The first aim of this paper is to prove that, for a large class of asset structures, $\mathcal{A}$ (and consequently $\mathcal{PE}$) is orientable if $S - J$ is even. More precisely, the asset structure V will be required to be smooth and transverse to the manifolds of low rank matrices[5]. For example, this last condition holds true for generic real asset structures, but also if the asset structure has full rank, i.e. if rank$V(p) = J$ for every $p \in P$.

The orientability of $\mathcal{A}$ easily implies the index theorem in the GEI model, which is the second contribution of our paper. A by-product is the recent result of Momi (see [12]), who has proved the index theorem in the GEI model for generic endowments, generic real asset structures and $S - J$ even. It is worth noting that one obtains the index theorem for an explicit class of asset structures (encompassing the class of real asset structures), and that our proof rests on a classical homotopy argument[6] (see [6] for a similar argument in complete

[4] $Z(w, p, E)$ is the aggregate excess demand of the economy obtained by substituting the abstract subspace E for the subspace of income transfers span$V(p)$ in the budget constraints of the agents. This modification allows to overcome the classical discontinuity of the GEI demand at prices p such that rank$V(p) < J$.

[5] This assumption has been introduced by Bottazzi (see [1]). Roughly, it requires that low rank price sets should not be too large.

[6] Momi's approach rests on Brown et al.'s paper [3], which provides a path following algo-

markets).

Our third contribution is to provide a natural definition of a regular economy in the GEI model, which generalizes Debreu's definition of a regular economy in the General Equilibrium with complete markets (GE) model (see [5]). Recall that a GE economy is regular if every equilibrium price is regular, in the sense that the matrix formed by deleting the last row and column from the jacobian matrix at the equilibrium price is nonsingular. We shall say that a GEI economy (parameterized by the endowment vector w) is regular if it is regular in the previous sense and if every pseudo-equilibrium price of the economy is an equilibrium, i.e., if for every (p, E) such that $(w, p, E) \in \mathcal{PE}$ one has $E = \text{span}V(p)$. We shall prove in this paper that almost all economies are regular, and our index theorem will be true for every regular economy.

The remainder of this paper is organized as follows. In Section 2, we state the main orientability result and some corollaries. In Section 3, we introduce the notion of regularity and we prove, as a first consequence of Section 2, that the index theorem holds true for regular economies and for $S - J$ even. In Section 4, we prove, as a second consequence of Section 2, that the pseudo-equilibrium manifold and the equilibrium manifold are orientable for $S - J$ even. Finally, the last section provides the proof of the orientability of the asset equilibrium manifold.

2 The main orientability result

2.1 The asset structure

We consider[7] in this paper the basic GEI model of an exchange economy with two periods $t = 0$ and $t = 1$, and K divisible goods available at each period

rithm for computing an equilibrium. This path following algorithm uses a family of homotopies allowing to overcome the standard discontinuity problem of the demand functions in the GEI model. More precisely, Momi's proof consists in relating the indices of these homotopies. Thus, any extension of Momi's paper (for example to multiperiod economies) would first require a similar extension of Brown et al's result.

[7]Throughout the text, all the manifolds considered are always assumed to be smooth and without boundary. By convention, a n-manifold with $n < 0$ is the empty set. If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ belong to $\mathbb{R}^n$, we denote by $x \cdot y = \sum_{i=1}^{n} x_i y_i$, the scalar product of $\mathbb{R}^n$. If $u_1, ..., u_k$ belong to E, a vector space, we denote by span$\{u_1, ..., u_k\}$ the vector subspace of E spanned by $u_1, ..., u_k$. If M is a matrix, we denote by span$M$ the vector subspace spanned by M. If f is a mapping from a set X to a set Y, then for every $X' \subset X$, $f\mid_{X'}$ denotes the restriction of f to X'. If J and S are two positive integers such that $J \leq S$, we denote by $G^J(\mathbb{R}^S)$ the set consisting of all linear subspaces of $\mathbb{R}^S$ of dimension J, called the (J-)Grassmannian manifold of $\mathbb{R}^S$. If f is a mapping differentiable at x, then we denote by $Df(x)$ the derivative of f at x. If M is a manifold, then for every $x \in M$, $T_x M$ denotes the tangent space of M at x. Finally, in all the paper, if M is a Banach manifold (resp. a finite dimensional Euclidean space), we say that a property P_λ, depending upon a parameter $\lambda \in M$, holds generically (or for generic $\lambda \in M$) if there exists an open and dense subset M' of M (resp. an open and full measure subset M' of M, for the Lesbegue measure on M) such that for every $\lambda \in M'$, P_λ is true.

$(K > 0)$.

The uncertainty in period $t = 1$ is represented by S states of nature ($S > 0$). Only one state happens at $t = 1$, and it is only known at the beginning of the period. For convenience, the unique state of nature (known with certainty) today (i.e., at $t = 0$) will be denoted $s = 0$. Hence, the number of commodities available either at $t = 0$ (with certainty) or at $t = 1$ (on each of the finite number S of possible states of nature) is $L := K(1 + S)$.

At each state $s = 0, 1, ..., S$, there is a spot market for each of the K physical goods. In addition, we assume that there exist at time $t = 0$ financial markets for a positive number J of assets ($1 \leq J \leq S$). Given the normalized price $p \in P := \{(p_1, ...p_L) \in \mathbb{R}^L_{++}, p_L = 1\}$ of the commodities, the asset j ($j = 1, ..., J$) can be bought at time $t = 0$ and delivers at time $t = 1$ a financial return $v_{s,j}(p)$ (in unit of account) if state s prevails. In the following, we denote by $V(p)$ the $S \times J$ matrix of returns at time $t = 1$, that is,

$$V(p) = (v_{s,j}(p))_{s=1,...,S, j=1,...,J}$$

Thus, if $\mathcal{M}(S \times J)$ denotes the set of all $S \times J$ matrices, then the asset structure is a mapping $V : P \to \mathcal{M}(S \times J)$. For every $\rho = 0, ..., J$, let us denote

$$\mathcal{M}^\rho(S \times J)) := \{M \in \mathcal{M}(S \times J), \text{rank} M = J - \rho\}.$$

We recall that it is a smooth submanifold of $\mathcal{M}(S \times J)$. In this paper we shall consider the broad class of *transverse asset structures*, defined as follows :

Definition 1 *The asset structure $V : P \to \mathcal{M}(S \times J)$ is called a transverse asset structure*[8] *if it is a smooth mapping and if :*
(T) Transversality Assumption *for every $\rho = 1, ..., J$, V is transverse to the manifold $\mathcal{M}^\rho(S \times J)$, which means that for every $\bar{p} \in P$ such that rank$V(\bar{p}) = J - \rho$, one has*

$$T_{V(\bar{p})}\mathcal{M}^\rho(S \times J) + DV(\bar{p})(T_{\bar{p}}P) = T_{V(\bar{p})}\mathcal{M}(S \times J).$$

Assumption (T) is a regularity assumption (see Appendix 6.2. for a definition in term of derivative). The interest of this condition is that it holds true for nominal asset structures, generic real asset structures, commodity forward contracts or more generally for generic smooth asset structures (see [1] p.66). Thus, it covers the cases of numerous classical asset structures.

2.2 The main result : orientability of the asset equilibrium space

The aim of this subsection is to prove that, under some conditions, the asset equilibrium space is an orientable manifold. This will play a crucial role in the

[8] This assumption has been introduced by Bottazzi ([1]), who has proved the existence of an equilibrium for generic endowments and for transverse asset structures.

proof of the index theorem in the next section, since it will allow us to apply oriented topological degree.

Now recall some important facts about orientability of a manifold and about topological degree, which will be usefull in the following.

First, recall that an orientation of a manifold M can be defined by an oriented atlas[9], i.e. an atlas of compatible charts of M (see Appendix 6.1. for more details). A manifold is orientable if it has an oriented atlas. For example, if a manifold M has an atlas with one chart (which means that M is diffeomorphic to an open subset of a Euclidean space), then it is clearly orientable.

Secondly, recall that if $f : X \to Y$ is a continuous and compactly rooted mapping[10] between orientable manifolds of the same dimension, then one can define $\deg(f)$, which denotes the standard topological degree of f (see [13] p.196). Notice that $\deg(f)$ depends on the orientation of X and Y. For example, if $f^{-1}(0) = \bar{x} \in X$ with f smooth at $\bar{x}$ and $Df(\bar{x})$ invertible, and if (ϕ, U) is a local chart of the oriented atlas of X at $\bar{x}$ (resp. (ϕ', U') is a local chart of the oriented atlas of Y at $f(\bar{x})$), then $\deg(f)$ is defined by[11] :

$$\deg(f) = \text{sign } \det D(\phi' \circ f \circ \phi^{-1})(\phi(\bar{x}))$$

and it is easy to see that it does not depend upon the choice of the local charts. Yet, it depends upon the choice of the oriented atlas.

We now fix the following orientation on P : consider the mapping

$$\phi : P \to \mathbb{R}^{L-1}$$

defined by

$$\phi(p_1, ..., p_{L-1}, 1) = (p_1, ..., p_{L-1})$$

for every $(p_1, ..., p_{L-1}, 1) \in P$. It is clearly a smooth diffeomorphism from P to $\mathbb{R}^{L-1}_{++}$. Thus $\{(\phi, P)\}$ is an atlas of P of compatible charts (with only one chart), which defines an orientation on P.

We now state the main orientability result of this paper.

Theorem 1 *If $V : P \to \mathcal{M}(S \times J)$ is a transverse asset structure, then :*

i) The asset equilibrium space

$$\mathcal{A} := \{(p, E) \in P \times G^J(\mathbb{R}^S),\ spanV(p) \subset E\}$$

is a smooth $(L-1)$-manifold.

ii) For every $\rho = 1, ..., J$

$$\mathcal{A}_\rho := \{(p, E) \in \mathcal{A}, \text{rank}V(p) = J - \rho\}$$

is either empty or a submanifold of $\mathcal{A}$ of codimension ρ^2.

iii) If $S - J$ is even or if rank$V(p) = J$ for every $p \in P$, then $\mathcal{A}$ is orientable.

[9]All the atlases considered throughout this paper will be smooth atlases.

[10]i.e. $f^{-1}(0)$ is a compact subset of X.

[11]In the following, the mapping $\phi' \circ f \circ \phi^{-1}$ is called the local representation of f in the charts (ϕ, U) and (ϕ', U').

Proof. The proof of Statements i) and ii) can be found in [2] (Proposition 4). Remark that the proof of Statement iii) is straightforward in the particular case when $\text{rank}V(p) = J$ for every $p \in P$. Indeed, in this case, the mapping $\pi \mid_{\mathcal{A}_0} = \pi \mid_{\mathcal{A}}$ is clearly a smooth diffeomorphism from $\mathcal{A}$ to the orientable manifold P. Consequently, since orientability is a diffeomorphism invariant, $\mathcal{A}$ is orientable. Now, for the general proof of Statement iii), see Section 4. □

3 The index theorem

3.1 The axiomatized GEI model

In this paper, we focuse attention on the notion of normalized no-arbitrage equilibrium (sometimes called effective equilibrium by some authors). This equilibrium notion is defined with respect to a single agent's present value price system, where the agent acts as if he were facing complete contingent markets, and it can be related to the standard notion of equilibrium (see [11]). We now provide an axiomatized definition of the economy[12], where the primitive concepts are the normalized no-arbitrage aggregate excess demand and the asset structure. In the following, let $\Omega = \mathbb{R}^L_{++}$. For every mapping $f = (f_1, ...f_L) : \mathbb{R}^L \to \mathbb{R}^L$, $\hat{f}$ denotes the mapping obtained by deleting the last component of f, i.e. $\hat{f} = (f_1, ..., f_{L-1})$.

The economy is characterized by :

1) The normalized no-arbitrage aggregate excess demand mapping Z, which is a smooth mapping

$$Z : \Omega \times P \times G^J(\mathbb{R}^S) \to \mathbb{R}^L.$$

Besides, one supposes that there exist two smooth mappings

$$Z^u : \Omega \times P \to \mathbb{R}^L$$

and

$$Z^c : P \times G^J(\mathbb{R}^S) \to \mathbb{R}^L$$

such that

$$Z(w, p, E) = Z^u(w, p) + Z^c(p, E)$$

for every $(w, p, E) \in \Omega \times P \times G^J(\mathbb{R}^S)$, and such that for every $w \in \Omega$:

[12] As in previous works (e.g., Chichilnisky and Heal (1996), Duffie and Shafer (1985)), we formalize the GEI model in an abstract fashion, only specifying the properties of the no-arbitrage aggregate excess demand mapping that suffices to derive our index theorem. In particular, the market span constraint at time $t = 1$ does not appear explicitly in this abstract definition.

i) (Walras Law) For every $(p, E) \in P \times G^J(\mathbb{R}^S)$, $p \cdot Z^c(p, E) = 0$ and $p \cdot Z^u_w(p) = 0$.[13]

ii) (Boundary condition) For every sequence $(p_\ell)_{\ell \in \mathbb{N}}$ of P converging to $\bar{p}$, $\bar{p} \notin \mathbb{R}^L_{++}$, one has $\lim_{\ell \to +\infty} \|Z^u_w(p_\ell)\| = +\infty$.

iii) (Bounded below) There exists $M \in \mathbb{R}$ such that for every $\ell = 1, ..., L$ and for every $(p, E) \in P \times G^J(\mathbb{R}^S)$, $Z^c(p, E) \cdot e_l \geq M$ and $Z^u_w(p) \cdot e_l \geq M$, where $\{e_1, ..., e_L\}$ denotes the canonical basis of $\mathbb{R}^L$.

iv) (Unconstrained agent) One has $(Z^u_w)^{-1}(0) = \{p(w)\}$, where $p(w) \in P$ satisfies

$$\text{sign } \det \partial_{(p_1,\dots,p_{L-1})} \hat{Z}^u_w(p(w)) = (-1)^{L-1} \tag{2}$$

Besides, for every $(p, w) \in P \times \Omega$

$$\text{rank} D_w \hat{Z}^u_w(p) = L - 1 \tag{3}$$

2) A transverse asset structure $V : P \to \mathcal{M}(S \times J)$.

The properties satisfied by $Z(w, p, E)$ can be derived from the definition of a normalized no-arbitrage equilibrium, where one agent is supposed to act as if he were facing complete contingent markets[14] and where E (in place of $\text{span}V(p)$) is supposed to be the subspace of income transfers of the agents at $t = 1$. Then, Z^u is the excess demand of the unconstrained agent (thus Z^u does not depend upon E), and Z^c is the aggregate excess demand of the remaining constraint agents. The set Ω, which will parameterized our economy, is the set of all possible intial endowments of the unconstrained agent. This explains why Z^c does not depend on w.

Moreover, Assumptions i), ii), iii) and iv) are true in the basic GEI model under the standard assumptions on preferences introduced by Debreu (see, for example, [8] p.292, Facts 4,5,6). The price vector $p(w)$ can be obtained as the normalized gradient at w of the unconstrained agent utility function, i.e., the present-value vector of the unconstrained agent at w. Equation 2 means that $p(w)$ is a regular equilibrium of the unconstrained agent's excess demand, for which the index theorem is true[15] (see below for a general definition of index). Besides, Equation 3 means that one can locally control $\hat{Z}^u$ by moving w.

[13]If $w \in \Omega$ is fixed, one denotes by $Z_w : P \times G^J(\mathbb{R}^S) \to \mathbb{R}^L$ and $Z^u_w : P \to \mathbb{R}^L$ the smooth mappings defined by $Z_w(p, E) = Z(w, p, E)$ and $Z^u_w(p) = Z^u(w, p)$ for every $(p, E) \in P \times G^J(\mathbb{R}^S)$.

[14]This assumption is usually referred to as Cass's trick. Roughly, it enables to yield an aggregate excess demand that blows up at the boundary of the price simplex. At an equilibrium, the unconstrained agent shares the same budget set as the other agents.

[15]The index of $\hat{Z}^u$ is $(-1)^{L-1} \det \partial_{(p_1,\dots,p_{L-1})} \hat{Z}^u_w(p(w)) = 1$.

Finally, notice that we shall often consider $\hat{Z}$ instead of Z, because from Walras' Law, the last coordinate of Z can be derived from the others coordinates of Z.

3.2 The notion of regularity

In the General Equilibrium with complete markets (GE) model, the index theorem holds true for every regular economy (see [5]). The aim of this subsection is to extend in a natural way the notion of regularity of Debreu to the GEI setting, in order to state the index theorem in the next section.

First recall that a pseudo-equilibrium of the economy $w \in \Omega$ is $(p, E) \in P \times G^J(\mathbb{R}^S)$ such that $Z(w, p, E) = 0$ and $\mathrm{span}V(p) \subset E$. Let us denote

$$\mathcal{PE} := \{(w, p, E) \in \Omega \times \mathcal{A}, Z(w, p, E) = 0\}$$

the set of all pseudo-equilibria, parameterized by the endowment vector w.

For every $w \in \Omega$, let us define

$$\mathcal{PE}'_w = \{p \in P, \exists E \in G^J(R^S), (w, p, E) \in \mathcal{PE}\}$$

be the set of all pseudo-equilibrium prices of the economy w.

Now, recall that an equilibrium of the economy $w \in \Omega$ is a price vector $p \in P$ such that $(w, p, \mathrm{span}V(p)) \in \mathcal{PE}$ (which requires $\mathrm{rank}V(p) = J$).[16] Let us denote

$$\mathcal{E} := \{(w, p) \in \Omega \times P, Z(w, p, \mathrm{span}V(p)) = 0, \mathrm{rank}V(p) = J\}$$

the set of all equilibria, parameterized by $w \in \Omega$. Lastly, for every $w \in \Omega$, we let

$$\mathcal{E}_w = \{p \in P, (w, p) \in \mathcal{E}\}.$$

We now define the notion of regular economy. To see that the following definition is relevant, notice that if p is an equilibrium, then from $\mathrm{rank}V(p) = J$ and from the smoothness of V and Z, the mapping $Z(p, \mathrm{span}V(p))$ is smooth on a neighborhood of p.

Definition 2 *One says that $p \in \mathcal{E}_w$ is a regular equilibrium of Z_w if*

$$\mathrm{rank}\ \partial_{(p_1,\ldots,p_{L-1})}\hat{Z}_w(p, \mathrm{span}V(p)) = L - 1.$$

The economy $w \in \Omega$ is said to be regular if every equilibrium of Z_w is regular and if $\mathcal{E}_w = \mathcal{PE}'_w$.

The following Proposition relates this definition to the mathematical notion of regularity.

[16] The usual definition of an equilibrium does not require $\mathrm{rank}V(p) = J$. But this is not a too strong requirement, since for transverse asset structures and for generic $w \in \Omega$, there does not exist equilibrium prices $p \in P$ such that $\mathrm{rank}V(p) < J$ (See [1]).

Proposition 1 *For every $\rho = 0, ..., J$, let $\mathcal{A}_\rho = \{(p, E) \in \mathcal{A}$, rank $V(p) = J - \rho\}$. The economy $w \in \Omega$ is regular if and only if for every $\rho = 0, ..., J$, 0 is a regular value of $\hat{Z}_w \mid_{\mathcal{A}_\rho}$.*[17]

Proof. We first claim that 0 is a regular value of $\hat{Z}_w \mid_{\mathcal{A}_0}$ if and only if every equilibrium of Z_w is regular. To prove this claim, let us define $\pi : P \times G^J(\mathbb{R}^S) \to P$ by $\pi(p, E) = p$ for every $(p, E) \in P \times G^J(\mathbb{R}^S)$. The mapping π is clearly a smooth diffeomorphism from $\mathcal{A}_0$ to $\pi(\mathcal{A}_0)$. Now, notice that

$$(\hat{Z}_w \mid_{\mathcal{A}_0})^{-1}(0) = \{(p, \text{span}V(p)), p \in \mathcal{E}_w\}.$$

Besides, the regularity of an equilibrium $p \in \mathcal{E}_w$ is equivalent to the invertibility of

$$\partial_{(p_1,...,p_{L-1})}(\hat{Z}_w \circ (\pi \mid_{\mathcal{A}_0})^{-1})(p),$$

and the regularity of $\hat{Z}_w \mid_{\mathcal{A}_0}$ at $(p,\text{span}V(p)) \in \mathcal{A}_0$ is equivalent to the invertibility of $D\hat{Z}_w \mid_{\mathcal{A}_0}$ at $(p,\text{span}V(p))$ (denoted $D(\hat{Z}_w \mid_{\mathcal{A}_0})(p,\text{span}V(p))$. Thus, the claim above is a clear consequence of

$$\partial_{(p_1,...,p_{L-1})}(\hat{Z}_w \circ (\pi \mid_{\mathcal{A}_0})^{-1})(p) = D(\hat{Z}_w \mid_{\mathcal{A}_0})(p, \text{span}V(p)) \circ \partial_{(p_1,...,p_{L-1})}(\pi \mid_{\mathcal{A}_0})^{-1}(p),$$

and of the invertibility of

$$\partial_{(p_1,...,p_{L-1})}(\pi \mid_{\mathcal{A}_0})^{-1}(p) = D(\pi \mid_{\mathcal{A}_0})^{-1}(p) \circ \phi^{-1}(p_1, ..., p_{L-1}),$$

where $\phi : P \to \mathbb{R}^{L-1}$, defined by $\phi(p_1, ..., p_{L-1}, 1) = (p_1, ..., p_{L-1})$ for every $(p_1, ..., p_L) \in P$, is a smooth diffeomorphism.

Now, to prove Proposition 1, from the claim above, it is sufficient to prove that $\mathcal{E}_w = \mathcal{PE}'_w$ if and only if for every $\rho = 1, ..., J$, 0 is a regular value of $\hat{Z}_w \mid_{\mathcal{A}_\rho}$. But from Statement ii) of Theorem 1, for every $\rho \geq 1$, $\mathcal{A}_\rho$ is a submanifold of $\mathcal{A}$ of codimension greater or equal to 1. Thus, if for every $\rho = 1, ..., J$, 0 is a regular value of $\hat{Z}_w \mid_{\mathcal{A}_\rho}$ then, from the implicit function theorem, $(\hat{Z}_w \mid_{\mathcal{A}_\rho})^{-1}(0)$ is empty, which implies $\mathcal{E}_w = \mathcal{PE}'_w$. Conversely, $\mathcal{E}_w = \mathcal{PE}'_w$ implies that for every $\rho = 1, ..., J$, $(\hat{Z}_w \mid_{\mathcal{A}_\rho})^{-1}(0)$ is empty, i.e. 0 is a regular value of $\hat{Z}_w \mid_{\mathcal{A}_\rho}$. □

Remark 1. In complete markets (i.e., if $J = S$ and rank$V(p) = S$ for every $p \in P$), the aggregate excess demand Z does not depend on span$V(p) = \mathbb{R}^S$, and one has $\mathcal{E}_w = \mathcal{PE}'_w$. Thus, the economy $w \in \Omega$ will be regular if for every equilibrium price $p \in P$, the jacobian $\partial_{(p_1,...,p_{L-1})}\hat{Z}_w(p)$ is invertible. This is the standard regularity condition of Debreu (see [5]), and it is known to be true for generic endowments. In the following, we prove that the regularity notion of Definition 2 holds true for generic endowments.

Theorem 2 *If V is a transverse asset structure, then for generic $w \in \Omega$, the economy w is regular.*

[17] Let M and N be two manifolds. We shall say that 0 is a regular value of a smooth mapping $f : M \to N$ (or equivalently that $f : M \to N$ is regular) if for every $x \in f^{-1}(0)$, $Df(x)$ in onto.

Proof. Suppose that V is a transverse asset structure. By assumption, for every $(p,w) \in P \times \Omega$, one has $\text{rank} D_w \hat{Z}^u_w(p) = L-1$. Consequently, for every $\rho = 0, ..., J$, one has $D_w \hat{Z} \mid_{\mathcal{A}_\rho} = L-1$. Thus, from Sard Theorem applied to $\hat{Z}_w \mid_{\mathcal{A}_\rho}$, for every $\rho = 0, ..., J$, there exists a generic subset $\Omega_\rho \subset \Omega$ such that for every $w \in \Omega_\rho$, 0 is a regular value of $\hat{Z}_w \mid_{\mathcal{A}_\rho}$. From Proposition 1, this implies that for every $w \in \cap_{\rho=0}^J \Omega_\rho$ (a generic subset of Ω), the economy w is regular. □

3.3 The index theorem

We are now ready to state and prove the index theorem.

Proposition 2 *If V is a transverse asset structure and if the economy $w \in \Omega$ is regular, then $\mathcal{E}_w$ is finite. For every regular economy $w \in \Omega$, the index of $\hat{Z}_w$ is the integer defined by*

$$\text{index}\hat{Z}_w = \sum_{p \in \mathcal{E}_w} (-1)^{L-1} \text{sign } \det \partial_{(p_1,\ldots,p_{L-1})} \hat{Z}_w(p, \text{span}V(p)).$$

Proof. To prove that $\mathcal{E}_w$ is finite if $w \in \Omega$ is regular, notice that one has

$$\{(p, \text{span}V(p)), p \in \mathcal{E}_w\} = (\hat{Z}_w \mid_{\mathcal{A}_0})^{-1}(0),$$

so that we only have to prove that $(\hat{Z}_w \mid_{\mathcal{A}_0})^{-1}(0)$ is finite.

From Proposition 1 and since $w \in \Omega$ is regular, 0 is a regular value of the mapping $\hat{Z}_w \mid_{\mathcal{A}_0}$. Thus, from the implicit function theorem, and since $\mathcal{A}_0$ is a $(L-1)$-manifold, the set $(\hat{Z}_w \mid_{\mathcal{A}_0})^{-1}(0)$ is a 0-submanifold of $\mathcal{A}_0$, i.e. a discrete subset of $\mathcal{A}_0$. Hence, it remains to prove that $(\hat{Z}_w \mid_{\mathcal{A}_0})^{-1}(0)$ is compact in $\mathcal{A}_0$.

Let $(p_n, E_n)_{n \in \mathbb{N}}$ be a sequence of $(\hat{Z}_w \mid_{\mathcal{A}_0})^{-1}(0)$, which means

$$(\hat{Z}_w \mid_{\mathcal{A}_0})(p_n, E_n) = 0 \tag{4}$$

and

$$\text{span}V(p_n) = E_n \tag{5}$$

From Equation 4, from the boundary condition satisfied by Z_w and from the compactness of $G^J(\mathbb{R}^S)$, one can suppose (up to an extraction) that the sequence $(p_n, E_n)_{n \in \mathbb{N}}$ converges to $(p, E) \in P \times G^J(\mathbb{R}^S)$. Besides, passing to the limit in Equation 4 and Equation 5, one clearly obtains $p \in \mathcal{PE'}_w$, this last subset being equal to $\mathcal{E}_w$ (from the definition of regularity of $w \in \Omega$).

Consequently one has $(p, E) \in (\hat{Z}_w \mid_{\mathcal{A}_0})^{-1}(0)$, which proves that $(\hat{Z}_w \mid_{\mathcal{A}_0})^{-1}(0)$ is a compact 0-submanifold of $\mathcal{A}_0$, i.e. a finite set. □

Theorem 3 *i) If V is a transverse asset structure and if $\mathcal{A}$ is orientable (which is true, for example, if $S-J$ is even or if $\text{rank}V(p) = J$ for every $p \in P$), then for generic endowments $w \in \Omega$, $\text{index}\hat{Z}_w = 1$.*

ii) If V is transverse then for generic endowments $w \in \Omega$, $\text{index}\hat{Z}_w = 1$ [modulo 2].

Proof. To prove Statement i), suppose that $\mathcal{A}$ is orientable.

First define an orientation of $\mathcal{A}_0$ as follows : let us consider the mapping

$$\pi : P \times G^J(\mathrm{I\!R}^S) \to P$$

defined by

$$\pi(p, E) = p$$

for every $(p, E) \in P \times G^J(\mathrm{I\!R}^S)$. Then it is clear that $\{(\phi \circ \pi, \mathcal{A}_0)\}$ is an oriented atlas of $\mathcal{A}_0$, which defined an orientation of $\mathcal{A}_0$.

Now, choose an orientation of $\mathcal{A}$ which is compatible with the orientation of $\mathcal{A}_0$.[18] For every $w \in \Omega$, let us define the mapping $H_w : [0,1] \times \mathcal{A} \to \mathrm{I\!R}^{L-1}$ by

$$H_w(t, p, E) = \hat{Z}^u_w(p) + t\hat{Z}^c(p, E)$$

for every $(p, E) \in \mathcal{A}$. From the boundary condition and since the mapping $\hat{Z}^u_w$ and $\hat{Z}^c$ are bounded below, the mapping H_w is clearly compactly rooted. Besides, it is a continuous homotopy from $\hat{Z}^u_w \circ \pi \mid_{\mathcal{A}}$ to $\hat{Z}_w \mid_{\mathcal{A}}$. Moreover, by assumption, $\mathcal{A}$ is orientable. Thus, one can apply oriented topological degree, and from the standard homotopy invariance property, one obtains :

$$\deg(\hat{Z}^u_w \circ \pi \mid_{\mathcal{A}}) = \deg(\hat{Z}_w \mid_{\mathcal{A}}) \tag{6}$$

The end of the proof of the index theorem consists in computing $\deg(\hat{Z}^u_w \circ \pi \mid_{\mathcal{A}})$ and in relating $\mathrm{index}\hat{Z}_w$ to $\deg(\hat{Z}_w \mid_{\mathcal{A}})$.

First, we claim that for generic $w \in \Omega$, $\mathrm{rank}V(p(w)) = J$, where $p(w)$ is the unique equilibrium price of $\hat{Z}^u_w$. Indeed, from Unconstrained Agent Assumption, 0 is a regular value of $\hat{Z}^u$. Hence, from Sard Theorem, for generic $w \in \Omega$, 0 is a regular value of $\hat{Z}^u_w$. Now, recalling that $P_\rho := \{p \in P,\ \mathrm{rank}V(p) = J - \rho\}$ $(\rho = 1, ..., J)$ is a $L - 1 - \rho(S - J + \rho)$ manifold (see, for example, [1]), the claim above follows from the implicit function theorem, since the latter implies that $(\hat{Z}^u_w \mid_{P_\rho})^{-1}(0)$ is empty for generic $w \in \Omega$ and for every $\rho = 1, ..., J$.

Consequently, from this claim and from Unconstrained Agent Assumption, for generic $w \in \Omega$ the mapping $\hat{Z}^u_w \circ \pi \mid_{\mathcal{A}}$ has a unique zero $(p(w), \mathrm{span}V(p(w))) \in \mathcal{A}_0$. Thus, using the local representation of $\hat{Z}^u_w \circ \pi \mid_{\mathcal{A}}$ in the chart $(\phi \circ \pi, \mathcal{A}_0)$ around $(p(w), \mathrm{span}V(p(w)))$, one can compute $\deg(\hat{Z}^u_w \circ \pi \mid_{\mathcal{A}})$ as follows :

$$\deg(\hat{Z}^u_w \circ \pi \mid_{\mathcal{A}}) = \text{sign det } \mathrm{D}(\hat{Z}^u_w \circ \pi \mid_{\mathcal{A}} \circ (\pi \mid_{\mathcal{A}_0})^{-1} \circ \phi^{-1})(p_1(w), ..., p_{L-1}(w)).$$

Finally, from Unconstrained Agent Assumption, one has :

$$\deg(\hat{Z}^u_w \circ \pi \mid_{\mathcal{A}}) = \text{sign det}\partial_{(p_1,...,p_{L-1})}\hat{Z}^u_w(p(w)) = (-1)^{L-1} \tag{7}$$

Now, let us relate $\mathrm{index}\hat{Z}_w$ to $\deg(\hat{Z}_w \mid_{\mathcal{A}})$. From Theorem 2, for generic w, the economy is regular, and in particular one has $\mathcal{E}_w = \mathcal{P}\mathcal{E}'_w$, i.e. $(\hat{Z}_w \mid_{\mathcal{A}})^{-1}(0)$ is

[18] which means that the orientation of $\mathcal{A}$, seen as a maximal oriented atlas of $\mathcal{A}$, must contain the global chart $(\phi \circ \pi, \mathcal{A}_0)$.

a finite subset of $\mathcal{A}_0$. Thus, using the local representation of $\hat{Z}_w \mid_{\mathcal{A}}$ in the chart $(\phi \circ \pi, \mathcal{A}_0)$ around every $p \in (\hat{Z}_w \mid_{\mathcal{A}})^{-1}(0)$, and from the additivity of degree, one obtains :

$$\deg(\hat{Z}_w \mid_{\mathcal{A}}) = \sum_{p \in \mathcal{E}_w} \text{sign det } D(\hat{Z}_w \circ (\pi \mid_{\mathcal{A}_0})^{-1} \circ \phi^{-1})(p_1, ..., p_{L-1}),$$

which can be rewritten :

$$\deg(\hat{Z}_w \mid_{\mathcal{A}}) = \sum_{p \in \mathcal{E}_w} \text{sign det } \partial_{p_1,...,p_{L-1}} \hat{Z}_w(p, \text{span}V(p)) = (-1)^{L-1} \text{ index} \hat{Z}_w \quad (8)$$

Now, Statement i) of Theorem 3 is a consequence of Equation 6, 7 and 8.

To prove Statement ii), remark that it is always possible to use modulo 2 degree in the arguments above (even if $\mathcal{A}$ is not orientable), which implies the (modulo 2) index theorem. □

Remark 2. Since smooth asset structures and real asset structures are generically transverse (see [1]), one obtains as a particular case the index theorem for generic endowments and generic smooth asset structures, and also for generic endowments and real asset structures (whose proof can be found in [12]). Above all, for the first time, one obtains the index theorem for several explicit (and non constant rank) asset structures, such as commodity forward contracts. Besides, for a given real asset structure (or more generally for a given smooth asset structure), it is possibly to check (at least theorically) if it is in the class of transverse asset structures, and so to check if the index theorem holds (for $S - J$ even).

4 Orientation of the equilibrium manifold

Several papers have been studied the structure of the pseudo-equilibrium manifold $\mathcal{PE}$ in the GEI model (e.g., [4], [14], [15]). If the pseudo-equilibrium manifold is parameterized by the endowments and a real asset structure, then it is proved in [14] that $\mathcal{PE}$ can be equipped with a manifold structure (more precisely with a vector bundle structure) and that $\mathcal{PE}$ is orientable, as a vector bundle, if and only if J is even. But recall that this does not imply that $\mathcal{PE}$ is orientable as a manifold. Yet, whether $\mathcal{PE}$ is orientable is an interesting question, since it would allow to use oriented topological degree for mappings defined on $\mathcal{PE}$.[19]

The purpose of this section is to prove, under the assumptions of Theorem 1, that the pseudo-equilibrium manifold and the equilibrium manifold are orientable manifolds, and to precise the structure of $\mathcal{E}$ in $\mathcal{PE}$.

[19] For example, consider the mapping $\Pi : \Omega \times P \times G^J(\mathbb{R}^S) \to \Omega$ defined by $\Pi(w, p, E) = w$ for every $(w, p, E) \in \Omega \times P \times G^J(\mathbb{R}^S)$. If $\mathcal{PE}$ was an oriented L-manifold, then, since Π is classicaly a smooth and proper mapping, one could apply oriented topological degree to $\Pi \mid_{\mathcal{PE}}$.

Theorem 4 *Let $V : P \to \mathcal{M}(S \times J)$ be a transverse asset structure. If $S - J$ is even or if rank$V(p) = J$ for every $p \in P$, then*

$$\mathcal{PE} := \{(w,p,E) \in \Omega \times \mathcal{A}, Z(w,p,E) = 0\}$$

is an orientable L-manifold.

Proof. From Walras' Law, one has

$$\mathcal{PE} = \{(w,p,E) \in \Omega \times \mathcal{A}, \hat{Z}(w,p,E) = 0\} = (\hat{Z} \mid_{\Omega \times \mathcal{A}})^{-1}(0) \tag{9}$$

Besides, from Equation 3, 0 is a regular value of $\hat{Z} \mid_{\Omega \times \mathcal{A}}$. Consequently, from the implicit function theorem and since $\Omega \times \mathcal{A}$ is a $(2L-1)$-manifold, $(\hat{Z} \mid_{\Omega \times \mathcal{A}})^{-1}(0)$ is a L-manifold.

Now, if $S - J$ is even or if rank$V(p) = J$ for every $p \in P$, then $\mathcal{A}$ is orientable (from Theorem 1), hence $\Omega \times \mathcal{A}$ is orientable.

To finish, recall that if $f : M \to \mathbb{R}^{L-1}$ is a smooth mapping with M an orientable smooth manifold, and if 0 is a regular value of f, then $f^{-1}(0)$ is an orientable submanifold of M (see [7] p.21, Theorem 2). Thus, applying this result to $\hat{Z} \mid_{\Omega \times \mathcal{A}}$, and from Equation 9, one obtains that $\mathcal{PE}$ is orientable. □

Theorem 5 *Let $V : P \to \mathcal{M}(S \times J)$ be a transverse asset structure.*

i) The equilibrium manifold $\mathcal{E} = \{(w,p,E) \in \mathcal{PE},$ rank$V(p) = J\}$ is an open and dense subset of $\mathcal{PE}$. Its complement is the disjoint union of a finite number of lower dimensional smooth submanifolds.

ii) If $S - J$ is even or if rank$V(p) = J$ for every $p \in P$, then the equilibrium manifold is an orientable L-manifold.

Proof. Since an open subset of an orientable manifold is an orientable manifold, Statement ii) is a consequence of Statement i) and of Theorem 4.

Now, notice that

$$\mathcal{PE} = (\cup_{\rho=1}^{J}\{(w,p,E) \in \mathcal{PE},\ \text{rank}V(p) = J - \rho\}) \cup \mathcal{E}.$$

Thus, proving Statement i) amounts to proving that for every $\rho = 1, ..., J$,

$$\{(w,p,E) \in \mathcal{PE},\ \text{rank}V(p) = J - \rho\}$$

is a lower dimensional submanifold of the L-manifold $\mathcal{PE}$. But one has

$$\{(w,p,E) \in \mathcal{PE}, \text{rank}V(p) = J - \rho\} = (\hat{Z} \mid_{\Omega \times \mathcal{A}_\rho})^{-1}(0),$$

where

$$\mathcal{A}_\rho = \{(p,E) \in \mathcal{A}, \text{rank}V(p) = J - \rho\}$$

is a strict submanifold of $\mathcal{A}$ (see Theorem 1). Hence, Statement i) is a consequence of the implicit function theorem and of the regularity of each mapping $\hat{Z} \mid_{\Omega \times \mathcal{A}_\rho}$, which is a consequence of Equation 3.□

Remark 3. Statement i) of Theorem 5 extends [14] (which treats the case of generic real asset structures) to the case of a given transverse asset structure.

5 Proof of Theorem 1

Let $V : P \to \mathcal{M}(S \times J)$ be a transverse asset structure and suppose that $S - J$ is even. Recall that for every $\rho \in [0, J]$, the set $\mathcal{A}_\rho$ is defined by

$$\mathcal{A}_\rho = \{(p, E) \in \mathcal{A} \mid \text{rank } V(p) = J - \rho\}.$$

Besides, let us recall that the orientation of the manifold $\mathcal{A}_0$ is fixed by its atlas $\{(\phi \circ \pi, \mathcal{A}_0)\}$ (with one global chart), where $\pi : P \times G^J(\mathbb{R}^S) \to P$ is defined by $\pi(p, E) = p$ and $\phi : P \to \mathbb{R}^{L-1}$ is defined by $\phi(p_1, ..., p_{L-1}, 1) = (p_1, ..., p_{L-1})$.

We separate the proof of Theorem 1 into three steps. The first step proves that around every element of $\mathcal{A}_1$, there exists a local chart of $\mathcal{A}$ compatible with the global chart of $\mathcal{A}_0$ defined above. Then, the second step proves that the global chart of $\mathcal{A}_0$ and the charts given by the first step constitute an oriented atlas of $\mathcal{A}_1 \cup \mathcal{A}_0$. Moreover, Step 3 proves that one does not change the orientability of a manifold by removing a finite number of submanifolds of codimension greater or equal to 2, which easily entails that $\mathcal{A}$ is orientable.

Now, let us begin the proof. First, note that if $J = S$, then $\mathcal{A}$ is diffeomorphic to P, and the proof of Statement iii) is straightforward. Thus, we can suppose that $J < S$.

Since $S - J$ is even and $1 \leq J < S$, one has $2 \leq S - J$ and $3 \leq S$. Besides, since $L = K(1 + S)$ and $1 \leq K$, one has $4 \leq L$.

The proof of Theorem 1 is a consequence of the three following steps.

Step one : for every $(\bar{p}, \bar{E}) \in \mathcal{A}_1$, there exists a chart $(\phi_{(\bar{p},\bar{E})}, U_{(\bar{p},\bar{E})})$ of $\mathcal{A}$ around $(\bar{p}, \bar{E})$, which is compatible with the chart $(\phi \circ \pi, \mathcal{A}_0)$.

To prove Step one, take $(\bar{p}, \bar{E}) \in \mathcal{A}_1$. First, we recall the parameterization of the $(L - 1)$-manifold $\mathcal{A}$ in a neighborhood of $(\bar{p}, \bar{E})$ (see [2] or Appendix 6.3. to have more details).

Without any loss of generality, up to a permutation of the rows and of the columns of $V(\bar{p})$, one can suppose that for every p in a neighborhood of $\bar{p}$,

$$V(p) = \begin{pmatrix} a(p) & b(p) \\ c(p) & d(p) \\ e(p) & f(p) \end{pmatrix} \tag{10}$$

where $a(p)$ is an invertible $(J - 1) \times (J - 1)$ matrix and $c(p)$ a $1 \times (J - 1)$ matrix.

Then there exists $\bar{C} \in \mathcal{M}((S - J) \times 1)$ and $U_{(\bar{p},\bar{C})}$, an open neighborhood of $(\bar{p}, \bar{C})$ in $P \times \mathcal{M}((S - J) \times 1)$, such that if one defines the mapping

$$\Phi : U_{(\bar{p},\bar{C})} \to P \times G^J(\mathbb{R}^S)$$

by

$$\Phi(p, C) = (p, \text{ span} \begin{pmatrix} I_{J-1} & 0 \\ 0 & 1 \\ -Cc(p)a^{-1}(p) + e(p)a^{-1}(p) & C \end{pmatrix}) \tag{11}$$

and the mapping $H : U_{(\bar{p},\bar{C})} \to \mathcal{M}((S-J)\times 1)$ by

$$H(p,C) := (f(p) - e(p)a^{-1}(p)b(p)) - C(d(p) - c(p)a^{-1}(p)b(p)) \qquad (12)$$

then one has[20] :
1) $\text{rank}\partial_p H(\bar{p},\bar{C}) = S - J$.
2) Φ is a smooth diffeomorphism from $U_{(\bar{p},\bar{C})}$ to $\Phi(U_{(\bar{p},\bar{C})})$.
3) $\Phi(H^{-1}(0))$ is an open neighborhood of $(\bar{p},\bar{E})$ in $\mathcal{A}$.

In the following, if $p = (p_1, ..., p_{L-1}, 1) \in P$, we shall denote $p' = (p_1, ..., p_{S-J})$ and $p'' = (p_{S-J+1}, ..., p_{L-1})$ (it is well defined because $S - J \geq 1$ and $S - J + 1 \leq L - 1$). Thus, one has $p = (p', p'', 1)$, and $H(p,C)$ can be written as a mapping of the 3 variables p', p'' and C.

In order to avoid some confusions in the following, $\partial_1 H(p,C)$, $\partial_2 H(p,C)$ and $\partial_3 H(p,C)$ will denote the derivative of H with respect to p', p'' and C, and $\partial_p H(p,C)$ will denote the derivative of H with respect to p.

Since $\text{rank}\partial_p H(\bar{p},\bar{C}) = S - J$, up to a permutation of the coordinates of p, one can suppose that

$$\text{rank}\partial_1 H(\bar{p},\bar{C}) = S - J, \qquad (13)$$

Thus, taking $U_{(\bar{p},\bar{C})}$ smaller if necessary, one can suppose :

"For every $(p,C) \in U_{(\bar{p},\bar{C})}, \det\partial_1 H(p,C)$ has a constant sign."

Besides, taking $U_{(\bar{p},\bar{C})}$ smaller again if necessary, the implicit function theorem says that there exists a smooth mapping g from a neighborhood U_1 of $(\bar{p}'',\bar{C})$ to $\mathbb{R}^{S-J}$ such that for every $(p,C) = ((p',p'',1),C) \in U_{(\bar{p},\bar{C})}$, the equation $H((p',p'',1),C) = 0$ is equivalent to $p' = g(p'',C)$.

Now, we define the mapping ψ from U_1 to $\mathcal{A}$ by

$$\forall (p'',C) \in U_1, \psi(p'',C) = \Phi((g(p'',C), p'', 1), C)$$

and we let $U_{(\bar{p},\bar{E})} = \psi(U_1)$ and $\phi_{(\bar{p},\bar{E})} = \psi^{-1}$. From the definition of Φ and g, $(\phi_{(\bar{p},\bar{E})}, U_{(\bar{p},\bar{E})})$ is clearly a chart of $\mathcal{A}$ around $(\bar{p},\bar{E})$.

To finish the proof of Step one, we have to check that the chart $(\phi_{(\bar{p},\bar{E})}, U_{(\bar{p},\bar{E})})$ is compatible with the chart $(\phi \circ \pi, \mathcal{A}_0)$, which amounts to proving that the determinant of the derivative of $\phi \circ \pi \circ \psi$ is positive on the set $\phi_{(\bar{p},\bar{E})}(\mathcal{A}_0 \cap U_{(\bar{p},\bar{E})})$.

[20] This parameterization can be explained as follows : since $\text{rank}V(\bar{p}) = J-1$, one can select $\tilde{V}$, a $S \times (J-1)$ submatrix of $V(p)$, which is full rank for every p on a neighborhood of $\bar{p}$. For every p on this neighborhood, the condition $\text{span}\tilde{V}(p) \subset E$ is equivalent to $E^{\perp} \in G^{S-J}(\text{span}\tilde{V}(p)^{\perp})$, which is a $(S-J)-$dimensional manifold. This is why such a subspace E can be parameterized by $C \in \mathcal{M}((S-J)\times 1)$ (the coordinate change of the parameterization $(p,C) \to (p,E)$ being given by Φ). Then, one can prove that the equation $\text{span}V(p) \subset E$ can be locally written $H(p,C) = 0$, where H is a regular mapping. See Appendix 6.3. for the detailed proof of 1), 2) and 3).

But for every $(p'', C) \in \phi_{(\bar{p},\bar{E})}(\mathcal{A}_0 \cap U_{(\bar{p},\bar{E})})$, one has $\phi \circ \pi \circ \psi(p'', C) = (g(p'', C), p'')$. Thus, one can compute

$$D(\phi \circ \pi \circ \psi)(p'', C) = \begin{pmatrix} \partial_{p''} g(p'', C) & \partial_C g(p'', C) \\ I_{L-1-S+J} & 0 \end{pmatrix} \tag{14}$$

Moreover, from the implicit function theorem and from the definition of g, one has

$$\partial_C g(p'', C) = -(\partial_1 H((g(p'', C), p'', 1), C))^{-1} \circ \partial_3 H((g(p'', C), p'', 1), C) \tag{15}$$

and from Equation 12, one has

$$\partial_3 H((g(p'', C), p'', 1), C) = \alpha((g(p'', C), p'', 1)) I_{S-J} \tag{16}$$

where $\alpha(p) := c(p)a^{-1}(p)b(p) - d(p)$ defines a continuous mapping from a neighborhood of $\bar{p}$ to $\mathbb{R}$. Consequently, from Equations 14, 15 and 16, one obtains, for every $(p'', C) \in \phi_{(\bar{p},\bar{E})}(\mathcal{A}_0 \cap U_{(\bar{p},\bar{E})})$:

$$\det D(\phi \circ \pi \circ \psi)(p'', C) = (\alpha((g(p'', C), p'', 1)))^{S-J} \times \det(\partial_1 H((g(p'', C), p'', 1), C))^{-1} \tag{17}$$

Since $S - J$ is even, the determinant above has a constant sign for every $(p'', C) \in \phi_{(\bar{p},\bar{E})}(\mathcal{A}_0 \cap U_{(\bar{p},\bar{E})})$. If it is negative, one can define a new chart of $\mathcal{A}$ around $(\bar{p}, \bar{E})$ by permuting the two first coordinates of the mapping $\phi_{(\bar{p},\bar{E})}$, which allows to obtain a positive determinant in Equation 17.

This ends the proof of Step 1.

Step two : $\mathcal{A}_0 \cup \mathcal{A}_1$ is orientable

Let w' be the atlas of $\mathcal{A}_0 \cup \mathcal{A}_1$ containing the chart $\{(\phi \circ \pi, \mathcal{A}_0)\}$ and the charts defined in Step 1. We shall prove that w' is an oriented atlas.

We have to prove that two charts in w' are compatible. By construction, since every chart in w' is compatible with the chart $\{(\phi \circ \pi, \mathcal{A}_0)\}$, we only have to prove that two charts in w' which are not equal to $\{(\phi \circ \pi, \mathcal{A}_0)\}$ are compatible.

Let $((p, E), (p', E'))$ in $\mathcal{A}_1 \times \mathcal{A}_1$, and let $(U_{(p,E)}, \phi_{(p,E)})$ and $(U_{(p',E')}, \phi_{(p',E')})$ be two local charts of $\mathcal{A}$ around (p, E) and (p', E'), as defined in Step one. Thus, $(\phi_{(p,E)}, U_{(p,E)})$ and $(\phi_{(p',E')}, U_{(p',E')})$ are compatible with $(\phi \circ \pi, \mathcal{A}_0)$.

We want to prove that for every $y \in \phi_{(p',E')}(U_{(p,E)} \cap U_{(p',E')})$, the determinant of $D(\phi_{(p,E)} \circ \phi^{-1}_{(p',E')})(y)$ is strictly positive. Suppose that it is false for some $y \in \phi_{(p',E')}(U_{(p,E)} \cap U_{(p',E')})$. Then, since $\phi_{(p,E)} \circ \phi^{-1}_{(p',E')}(y)$ is invertible, one has

$$\det D(\phi_{(p,E)} \circ \phi^{-1}_{(p',E')})(y) < 0. \tag{18}$$

Now, from Theorem 1, $\mathcal{A}_0$ is an open and dense subset of $\mathcal{A}$. Consequently, by continuity of $\mathrm{D}(\phi_{(p,E)} \circ \phi^{-1}_{(p',E')})$ and since $\phi_{(p',E')}(U_{(p,E)} \cap U_{(p',E')})$ is open, one can perturb y in order to have $\det D(\phi_{(p,E)} \circ \phi^{-1}_{(p',E')})(y) < 0$ and

$$y \in \phi_{(p',E')}(U_{(p,E)} \cap U_{(p',E')} \cap \mathcal{A}_0).$$

Hence, one can write

$$D(\phi_{(p,E)} \circ \phi^{-1}_{(p',E')})(y) = D(\phi_{(p,E)} \circ (\phi \circ \pi)^{-1}) \circ ((\phi \circ \pi) \circ \phi^{-1}_{(p',E')})(y) \qquad (19)$$

But, by assumption, $\phi\circ\pi$ is compatible with $\phi_{(p,E)}$ and with $\phi_{(p',E')}$, which implies, with Equation 19 :

$$\det D(\phi_{(p,E)} \circ \phi^{-1}_{(p',E')})(y) > 0 \qquad (20)$$

But Inequations 18 and 20 cannot hold together, which finally proves the compatibility condition, and ends the proof of Step two.

Step three : $\mathcal{A}$ is orientable

With the notations of Theorem 1, one has

$$\mathcal{A} = (\cup_{\rho=2}^{J} \mathcal{A}_\rho) \cup (\mathcal{A}_0 \cup \mathcal{A}_1) \qquad (21)$$

where for every $\rho = 2, ..., J$, $\mathcal{A}_\rho$ is a submanifold of $\mathcal{A}$ of codimension greater or equal to 2.

Besides, it is well known that a manifold is orientable if and only if orientation is preserved moving along every loop in this manifold (see Appendix 6.1.). Thus, if one supposes that $\mathcal{A}$ is not orientable, then there exists a continuous mapping $\lambda : [0, 1] \to \mathcal{A}$, with $\lambda(0) = \lambda(1)$, which reverses orientation.

Now, from standard transversality arguments, since the $\mathcal{A}_\rho$ are of codimension greater or equal to 2 ($\rho = 2, ..., k$) and from Equation 21, one can perturb λ, within its homotopy class, to obtain a loop in $\mathcal{A}_0 \cup \mathcal{A}_1$ which reverses orientation[21]. Thus, using the same characterization of orientability applied to $\mathcal{A}_0 \cup \mathcal{A}_1$, one obtains a contradiction with the orientability of $\mathcal{A}_0 \cup \mathcal{A}_1$.

6 Appendix

6.1 Orientation of a manifold

First recall the definition of an orientable manifold. Consider two charts $\{\phi_1, U_1\}$ and $\{\phi_2, U_2\}$ of a smooth, boundaryless n-manifold M, which means that U_1 and U_2 are open subsets of M and that ϕ_1 (resp. ϕ_2) is a smooth diffeomorphism from U_1 (resp. U_2) to an open neighborhood of $\mathbb{R}^n$. These two charts are said to be compatible if the coordinate change $\phi_2 \circ \phi_1^{-1}$ has a positive Jacobian determinant at every point $x \in \phi_1(U_1 \cap U_2)$. The manifold M is said to be orientable if it has an atlas of compatible charts, called an oriented atlas. A maximal atlas of this kind is called an orientation of M. Remark that every oriented atlas defines an orientation, since an oriented atlas is always included in a maximal oriented atlas. Besides, if M is orientable, it is easy to see that there are two

[21]The property of reversing or preserving orientation is homotopy invariant. See [10] p.104.

possible orientations, often denoted w and $-w$. Then, every chart of M must belong to w or to $-w$.

For example, a n-dimensional vector space E is orientable, and clearly, any basis $\{e_1, ..., e_n\}$ of E allows to define an orientation of E. Indeed, a global chart $\phi : E \to \mathbb{R}^n$ can be defined by $\phi(x_1 e_1 + ... + x_n e_n) = (x_1, ..., x_n)$, and $\{(\phi, E)\}$ is an atlas of compatible charts of E. We let the reader check that two basis $\{e_1, ..., e_n\}$ and $\{f_1, ..., f_n\}$ of E define the same orientation if and only if the determinant of $A : E \to E$ such that $A(e_i) = f_i$ for every i is positive.

Let M be a manifold (possibly non orientable), and let us consider a continuous path in M, i.e. a continuous mapping $\lambda : [0, 1] \to M$. Let w be an orientation of the vector space $T_{\lambda(0)}M$. Then, it is possible to propagate the orientation w (by moving along λ) to obtain an orientation of $T_{\lambda(1)}M$ (see [10] p.104). We denote by $\lambda^*(w)$ such an orientation of $T_{\lambda(1)}M$. One says that a loop λ (i.e. a continous path such that $\lambda(0) = \lambda(1)$) preserves orientation if $\lambda^*(w) = w$.

In this paper, one uses the following characterization of orientability :

Theorem *M is orientable if and only if every loop $\lambda : [0, 1] \to M$ preserves orientation.*

Proof. See [10] p.104.

6.2 The transversality assumption

Let $\bar{p} \in P$ such that rank $V(\bar{p}) = J - \rho$ for some $\rho \in \{1, ..., J\}$. Without any loss of generality, up to a permutation of the rows and of the columns of $V(\bar{p})$, one can suppose that for every p in a neighborhood $U \subset P$ of $\bar{p}$,

$$V(p) = \begin{pmatrix} a(p) & b(p) \\ c(p) & d(p) \end{pmatrix} \tag{22}$$

where $a(p)$ is a $(J - \rho) \times (J - \rho)$ invertible matrix.

Lemma 1 *The transversality condition (T) is equivalent to the regularity of the mapping $f : U \to \mathcal{M}((S - J + \rho) \times \rho)$ defined by $f(p) = d(p) - c(p)a^{-1}(p)b(p)$ for every $p \in U$. Besides, $f^{-1}(0) \cap U = \{p \in U, \text{rank } V(p) = J - \rho\}$.*

Proof. The proof can be found in [2]. See also [1] for the same statement.

6.3 Parameterization of $\mathcal{A}$ at $(\bar{p}, \bar{E}) \in \mathcal{A}_1$.

First, up to a permutation of the rows and of the columns of $V(\bar{p})$, one can suppose that for every p in a neighborhood $U_{\bar{p}} \subset P$ of $\bar{p}$,

$$V(p) = \begin{pmatrix} a(p) & b(p) \\ c(p) & d(p) \\ e(p) & f(p) \end{pmatrix} \tag{23}$$

where $a(p)$ is an invertible $(J-1) \times (J-1)$ matrix and $c(p)$ a $1 \times (J-1)$ matrix. Consequently, applying Lemma 1, the condition $\text{rank}V(p) = J - 1$ is equivalent, on $U_{\bar{p}}$, to the following regular equation (the equation $F(p) = 0$ is said to be regular if 0 is a regular value of F) :

$$\begin{pmatrix} d(p) \\ f(p) \end{pmatrix} - \begin{pmatrix} c(p)a^{-1}(p)b(p) \\ e(p)a^{-1}(p)b(p) \end{pmatrix} = 0 \tag{24}$$

Moreover, the condition $\text{span}V(\bar{p}) \subset \bar{E}$ means exactly that there exists a $S \times 1$ matrix $\begin{pmatrix} b_1 \\ d_1 \\ f_1 \end{pmatrix}$, where b_1 is a $(J-1) \times 1$ matrix and d_1 is a 1×1 matrix, such that

$$\bar{E} = \text{ span} \begin{pmatrix} a(\bar{p}) & b_1 \\ c(\bar{p}) & d_1 \\ e(\bar{p}) & f_1 \end{pmatrix} \tag{25}$$

Without any loss of generality one may suppose that the first $J \times J$-submatrix of this last matrix is invertible, wich implies that $Y(\bar{p}) := c(\bar{p})a^{-1}(\bar{p})b_1 - d_1$ is invertible. Then, multiplying the matrix of the last equation by the two $J \times J$ invertible matrices

$$\begin{pmatrix} a^{-1}(\bar{p}) & a^{-1}(\bar{p})b_1 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} I_{J-1} & 0 \\ -Y(\bar{p})^{-1}c(\bar{p})a^{-1}(\bar{p}) & Y(\bar{p})^{-1} \end{pmatrix},$$

one obtains

$$\bar{E} = \text{ span} \begin{pmatrix} I_{J-1} & 0 \\ 0 & 1 \\ -\bar{C}c(\bar{p})a^{-1}(\bar{p}) + e(\bar{p})a^{-1}(\bar{p}) & \bar{C} \end{pmatrix} \tag{26}$$

for some $(S-J) \times 1$ matrix $\bar{C}$. Now, recall that if $\bar{E} = \text{ span} \begin{pmatrix} I_J \\ \bar{A} \end{pmatrix}$ where $\bar{A}$ is a $(S-J) \times J$ matrix, then every $J-$subspace E of $\mathbb{R}^S$ in a neighborhood of $\bar{E}$ can be written $\bar{E} = \text{ span} \begin{pmatrix} I_J \\ A \end{pmatrix}$ for a unique $(S-J) \times J$ matrix A, and it allows to define a local chart from a neighborhood of $\mathcal{M}((S-J) \times J)$ to a neighborhood of $\bar{E}$.

Besides, the construction that leads to Equation 26 can be done for every (p, E) in a neighborhood of $(\bar{p}, \bar{E})$ and such that $\text{span}V(p) \subset E$, so that E can be univocally and smoothly parametrized by a $(S-J) \times 1$ matrix C and $p \in P$, with

$$E = \Psi(p, C) := \text{ span} \begin{pmatrix} I_{J-1} & 0 \\ 0 & 1 \\ -Cc(p)a^{-1}(p) + e(p)a^{-1}(p) & C \end{pmatrix} \tag{27}$$

Thus, on a neighborhood $U^1_{(\bar{p},\bar{E})}$ of $(\bar{p},\bar{E})$ (in $P \times G^J(\mathbb{R}^S)$), the equation span$V(p) \subset E$ is equivalent to

$$\text{rank}\begin{pmatrix} I_{J-1} & 0 & b(p) \\ 0 & 1 & d(p) \\ -Cc(p)a^{-1}(p) + e(p)a^{-1}(p) & C & f(p) \end{pmatrix} = J \tag{28}$$

where (p,C) belongs to a neighborood $U_{(\bar{p},\bar{C})}$ of $(\bar{p},\bar{C})$. From Lemma 1, this is finally equivalent to $f(p) + Cc(p)a^{-1}(p)b(p) - e(p)a^{-1}(p)b(p) - Cd(p) = 0$, i.e. to

$$H(p,C) := (f(p) - e(p)a^{-1}(p)b(p)) - C(d(p) - c(p)a^{-1}(p)b(p)) = 0 \tag{29}$$

where H is a mapping from $U_{(\bar{p},\bar{C})}$ to $\mathcal{M}((S-J)\times 1)$. Besides, since Equation (24) is regular, 0 is a regular value of H.

Let us define $\Phi : U_{(\bar{p},\bar{C})} \to P \times G^J(\mathbb{R}^S)$ by

$$\Phi(p,C) = (p, \Psi(p,C))$$

for every $(p,C) \in U_{(\bar{p},\bar{C})}$. It is a smooth diffeomorphism from $U_{(\bar{p},\bar{C})}$ to $\Phi(U_{(\bar{p},\bar{C})})$. Finally, we proved that $\Phi(H^{-1}(0))$ is an open neighborhood of $(\bar{p},\bar{E}) \in \mathcal{A}_1$ in $\mathcal{A}$. Moreover, we have rank$\partial_p H(\bar{p},\bar{C}) = S - J$, which is a clear consequence of the regularity of Equation 24.

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